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Measuring Rényi dimensions by a modified box algorithm

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Abstract. Two different origins of statistical errors in multifractal analysis by the box algorithm are investigated. We propose a modified box algorithm reducing the statistical errors and allowing a more accurate estimation of the region where power law scaling is present.

1. Introduction

Strange attractors play an important role in investigations of chaotic motion. Their geometrical properties which are related to the dynamics of the described system [1–3] are characterized by their $f(\alpha)$-spectrum which is obtained from a spectrum of exponents, the Rényi-dimensions D_q $(q \in \mathbb{R})$ [1, 4].

The D_q are usually measured by scanning a finite time series. The question of how to extract as much information about the Rényi dimensions as possible from the data set leads to the problem of the precision of measured D_q. Holzfuss and Mayer-Kress [5], who introduced an error estimation for the measurement of some exponents from time series, demonstrated that existing errors are usually underestimated. Here we will give an alternative procedure to estimate the errors and give expressions for the fluctuations.

The paper is organized as follows. In section 2 we give a short introduction to the partition function formalism and the box counting algorithm. In sections 3 and 4 the fluctuations in the box algorithm due to the finite length of time series are investigated for positive and negative q respectively. In section 5 the box algorithm is tested at a sample multifractal, the generalized Baker's map, and it is shown that the algorithm itself causes additional fluctuations. In section 6 a modified algorithm is introduced which reduces those additional fluctuations and allows a more precise estimation of the D_q.

2. The box algorithm

On an attractor A a measure μ is given by the point-density (of an infinitely long time series) which should be normalized, so that

$$\mu(A) = 1 \tag{1}$$

holds. The attractor in the box algorithm is covered by not more than countable many disjoint sets s_i with radius $l_i < l$ and the partition function is defined as

$$\Gamma(q, D, l, \{l_i\}) = \sum_i \frac{p_i^q}{l_i^{(q-1)D}} \tag{2}$$

with

$$p_i = \int_{s_i \cap A} \mathrm{d}\mu \qquad D \geq 0.$$

The optimization over all possible coverings leads to

$$\Gamma(q, D, l) = \sup \Gamma(q, D, l, \{l_i\}) \tag{3}$$

for $q > 1$ and

$$\Gamma(q, D, l) = \inf \Gamma(q, D, l, \{l_i\}) \tag{4}$$

for $q < 1$. If we consider the limit of small diameters ($l \to 0$) we get

$$\Gamma(q, D) = \lim_{l \to 0} \Gamma(q, D, l). \tag{5}$$

The Rényi dimension of degree q is then defined as

$$D_q = \inf_{\Gamma(q, D) = 0} D \tag{6}$$

if $q > 1$ and

$$D_q = \sup_{\Gamma(q, D) = 0} D \tag{7}$$

if $q < 1$. A detailed description of the partition function formalism can be found in [6, 7].

Using the box algorithm, the attractor is covered by a grid of boxes b_i, $i = 1, 2, \ldots,$ of size δ. The moments χ^q of the probability distribution p_i are defined as

$$\chi^q := \sum_i p_i^q \tag{8}$$

where p_i is $p_i = \mu(A \cap b_i)$. The χ^q depend on δ and the D_q are

$$D_q := \lim_{\delta \to 0} \frac{1}{q-1} \frac{\partial \log \chi^q(\delta)}{\partial \log \delta} \tag{9}$$

for $q \neq 1$ and [8]

$$D_1 = \lim_{q \to 1} D_q. \tag{10}$$

The box size δ here plays the role of arguments l and l_i of the partition function. Instead of the optimization (3) and (4) the box algorithm requires a cover of equal-sized boxes with a fixed place in the used lattice. The moments $\chi^q(\delta)$ then in some way are approximations of $\Gamma(q, D, l = \delta) \cdot \delta^{(q-1)D}$.

To evaluate the Rényi dimensions by the box algorithm the used time series should be long enough that first it reaches every part of the attractor and second the correlation between the points becomes negligible and their positions are governed just by the attractor's point density. Then the measuring can be regarded as a random experiment and the probability that the ith box contains $k_i = k$ points out of a time series of n points is

$$P_n(k_i = k) = \binom{n}{k} p_i^k (1-p_i)^{n-k}. \tag{11}$$

This leads, if one regards the contents of the boxes as independent, to the expectation value of χ^q

$$\langle \chi_n^q \rangle = \sum_i \sum_{k=1}^{n} \binom{n}{k} p_i^k (1-p_i)^{n-k} \left(\frac{k}{n}\right)^q. \tag{12}$$

To regard the contents of the boxes as independent means that each box is filled by its own time series of n points or, in other words, the constraint $\Sigma_i k_i = n$ does not hold and n is not a constant, but a random variable whose expectation is the number of points in the time series and whose standard deviation is $\sqrt{n(1-\Sigma p_i^2)}$, so that the relative width of its distribution becomes small for large n. Under these conditions (12) will be a good approximation for the values of the moments.

To measure the D_q, the $\chi^q(\delta)$ are evaluated for different $\delta(\delta = \delta_1, \delta_2, \ldots, \delta_N)$. A least-squares fit is used to draw a straight line through the points $(\log \delta_i, \log \chi^q(\delta_i))$, $i = 1, \ldots, N$, whose slope is $(q-1)D_q$. Here we assume that $\chi^q(\delta)$ can be described by a power law which at least is fulfilled for a finite range of δ. This range of δ-values has to be detected very carefully.

A least-squares fit of a straight line through a set of points (x_i, y_i), $i = 1, \ldots, N$, leads to a slope S of

$$S = \frac{N \Sigma_{i=1}^N x_i y_i - \Sigma_{i=1}^N x_i \Sigma_{j=1}^N y_j}{N \Sigma_{i=1}^N x_i^2 - [\Sigma_{i=1}^N x_i]^2}. \tag{13}$$

In our case we assume that the x_i are exact and thus we can write for the error of S

$$\Delta S \approx \sum_{i=1}^{N} \left|\frac{\partial S}{\partial y_i}\right| \Delta y_i = \frac{N \Sigma_{i=1}^N |x_i - \bar{x}| \Delta y_i}{N \Sigma_{i=1}^N x_i^2 - [\Sigma_{i=1}^N x_i]^2}. \tag{14}$$

So we can determine $(q-1)D_q$ by equation (13) and, if we know the Δy_i, the corresponding errors. For analytic expressions of the errors made in this measurements see the following section.

3. Positive values of q

To extract the expectation values of the moments $\langle \chi_n^q \rangle$ scanning a time series of length n we define the quantity

$$m_n^q(p) := \sum_{k=1}^{n} \binom{n}{k} p^k (1-p)^{n-k} k^q. \tag{15}$$

The expectation values of the moments are then given by

$$\langle \chi_n^q \rangle = \frac{1}{n^q} \sum_i m_n^q(p_i). \tag{16}$$

The $\langle \chi_n^q \rangle$ are continuous and monotonic functions of q. It can be easily seen that

$$m_n^{q+1}(p) = p(1-p)^{n+1} \frac{\partial[(1-p)^{-n} m_n^q(p)]}{\partial p} \tag{17}$$

and from $m_n^0(p)=1-(1-p)^n$ one obtains

$$m_n^1(p)=np \tag{18}$$

$$m_n^2(p)=(n^2-n)p^2+np \tag{19}$$

$$m_n^3(p)=(n^3-3n^2+2n)p^3+(3n^2-3n)p^2+np \tag{20}$$

$$m_n^4(p)=(n^4-6n^3+11n^2-6n)p^4+(6n^3-18n^2+12n)p^3+(7n^2-7n)p^2+np \tag{21}$$

The expectation values of the corresponding moments are given by

$$\begin{aligned}\langle\chi_n^0\rangle&=\sum_i[1-(1-p_i)^n]\\&=\chi^0-\sum_i(1-p_i)^n\\&=\chi^0+C_n^0\end{aligned} \tag{22}$$

$$\begin{aligned}\langle\chi_n^2\rangle&=\left(1-\frac{1}{n}\right)\chi^2+\frac{1}{n}\\&=\left(1-\frac{1}{n}\right)\chi^2+C_n^2\end{aligned} \tag{23}$$

$$\begin{aligned}\langle\chi_n^3\rangle&=\left(1-\frac{3}{n}+\frac{2}{n^2}\right)\chi^3+\left(\frac{3}{n}-\frac{3}{n^2}\right)\chi^2+\frac{1}{n^2}\\&=\left(1-\frac{3}{n}+\frac{2}{n^2}\right)\chi^3+C_n^3\end{aligned} \tag{24}$$

$$\begin{aligned}\langle\chi_n^4\rangle&=\left(1-\frac{6}{n}+\frac{11}{n^2}-\frac{6}{n^3}\right)\chi^4+\left(\frac{6}{n}-\frac{18}{n^2}+\frac{12}{n^3}\right)\chi^3+\left(\frac{7}{n^2}-\frac{7}{n^3}\right)\chi^2+\frac{1}{n^3}\\&=\left(1-\frac{6}{n}+\frac{11}{n^2}-\frac{6}{n^3}\right)\chi^4+C_n^4\\&\dots\end{aligned} \tag{25}$$

with

$$C_n^0=-\sum_i(1-p_i)^n$$

$$C_n^2=\frac{1}{n}$$

$$C_n^3=\chi^2\left(\frac{3}{n}-\frac{3}{n^2}\right)+\frac{1}{n^2}$$

$$C_n^4=\chi^3\left(\frac{6}{n}-\frac{18}{n^2}+\frac{12}{n^3}\right)+\chi^2\left(\frac{7}{n^2}-\frac{7}{n^3}\right)+\frac{1}{n^3}.$$

The constant prefactor before the terms χ^q in the expressions for the expectation values (22–25) will disappear in the logarithmic derivation, but the C_n^q are corrections and n

must be chosen large enough so that they become sufficiently small. The variance and the standard deviation are given by

$$\Delta^2(\chi^q)=\frac{1}{n^{2q}}\sum_i D^2(k_i^q)$$

$$=\frac{1}{n^{2q}}\sum_i [m_n^{2q}(p_i)-(m_n^q(p_i))^2]$$

$$\Rightarrow\Delta\chi^q=\sqrt{\frac{1}{n^{2q}}\sum_i [m_n^{2q}(p_i)-(m_n^q(p_i))^2]} \tag{26}$$

which are continuous functions of q which can be interpolated between the integer values of q. Note that the boxes are independent in this counting. For $q=0$ and $q=2$ relation (26) gives

$$\Delta\chi^0=\sqrt{\sum_i [(1-p_i)^n-(1-p_i)^{2n}]} \tag{27}$$

and

$$\Delta\chi^2=\sqrt{\sum_i\left[p_i^4\left(\frac{-4}{n}+\frac{10}{n^2}-\frac{6}{n^3}\right)+p_i^3\left(\frac{4}{n}-\frac{16}{n^2}+\frac{12}{n^3}\right)+p_i^2\left(\frac{6}{n^2}-\frac{7}{n^3}\right)+\frac{p_i}{n^3}\right]} \tag{28}$$

respectively. For sufficiently large n in (28) the term $\Sigma\, p_i^3(4/n)$ dominates in the square root and the error of χ^2 shows a $\sqrt{1/n}$ behaviour which corresponds to a similar result of Theiler [9] who found the same behaviour in the correlation algorithm.

4. Negative values of q

To get the expectation values of the moments for negative q one has to integrate (17). This yields

$$m_n^q(p)=(1-p)^n\int\frac{m_n^{q+1}(p)}{p(1-p)^{n+1}}\,\mathrm{d}p \tag{29}$$

which is for $q=-1$

$$m_n^{-1}(p)=(1-p)^n\int_{p_0}^{p}\frac{(1-s)^{-n-1}-(1-s)^{-1}}{s}\,\mathrm{d}s+m_n^{-1}(p_0). \tag{30}$$

Using

$$\frac{1}{p}=(1-p)^n\frac{1}{p}(1-p)^{-n}$$

$$=(1-p)^n\int_{p_0}^{p}\left[\frac{-1}{s^2(1-s)^n}+\frac{n}{s(1-s)^{n+1}}\right]\mathrm{d}s+\frac{1}{p_0} \tag{31}$$

and choosing $p_0=\frac{1}{2}$, since we then can evaluate the integration constant

$$m_n^{-1}(p_0)=m_n^{-1}(\tfrac{1}{2})\approx\frac{2}{n^2} \tag{32}$$

we can write

$$|C_n^{-1}|=\left|m_n^{-1}(p)-\frac{1}{np}\right|$$

$$\leqslant(1-p)^n\left|\int_1^p\frac{\mathrm{d}s}{s(1-s)}\right|+\frac{(1-p)^n}{n}\left|\int_1^p\frac{\mathrm{d}s}{s^2(1-s)^n}\right|+\frac{2}{n^2}$$

$$\leqslant(1-p)^n\log\frac{1-p}{p}+\frac{1}{(1-p)p^2n^2}. \tag{33}$$

To obtain the variance of χ^{-1} we have to evaluate m_n^{-2} (compare equation (26)) which is by (29)

$$m_n^{-2}(p)=(1-p)^n\int_{p_0}^p\frac{m_n^{-1}(\bar p)\,\mathrm{d}\bar p}{(1-\bar p)^{n+1}\bar p}+m_n^{-2}(p_0)$$

$$=(1-p)^n\int_{p_0}^p\left\{\frac{1}{(1-\bar p)^{n+1}\bar p^2n}-\frac{1}{(1-\bar p)\bar p}\int_{\bar p_0}^{\bar p}\frac{\mathrm{d}s}{s(1-s)}\right.$$

$$\left.+\frac{1}{n(1-\bar p)\bar p}\int_{\bar p_0}^{\bar p}\frac{\mathrm{d}s}{s^2(1-s)^n}+\frac{m_n^{-1}(\bar p_0)}{(1-\bar p)^{n+1}\bar p}\right\}\mathrm{d}\bar p+m_n^{-2}(p_0). \tag{34}$$

We use

$$\frac{1}{p^2}=(1-p)^n\int_{p_0}^p\left[\frac{n}{(1-\bar p)^{n+1}\bar p^2}-\frac{1}{(1-\bar p)\bar p^3}\right]\mathrm{d}\bar p+\frac{1}{p_0^2} \tag{35}$$

and set $p_0=\bar p_0=\frac{1}{2}$; we then can evaluate the integration constant to

$$m_n(\bar p_0)=m_n^{-2}(\tfrac{1}{2})-\frac{4}{n^2}\approx\frac{16}{n^3} \tag{36}$$

and obtain

$$|C_n^{-2}|=\left|m_n^{-2}-\frac{1}{n^2p^2}\right|$$

$$\leqslant(1-p)^n\left(\log\frac{1-p}{p}+\frac{1}{n^2p^3}\right)+\frac{2}{n^2p}\left(1-\frac{1}{n}\right)+\frac{16}{n^3}. \tag{37}$$

The variance of χ^{-1} is now

$$\Delta^2(\chi^{-1})=n^2\sum_i\left[m_n^{-2}(p_i)-(m_n^{-1}(p_i))^2\right]$$

$$\leqslant n^2\sum_i\left[\frac{(1-p_i)^n}{n^2p_i^3}+\frac{2-(2/n)}{n^2p_i}+(1-p_i)^{2n}\log^2\frac{1-p_i}{p_i}\right.$$

$$\left.+\frac{1}{(1-p_i)^2p_i^4n^4}+\frac{2(1-p_i)^{n-1}\log[(1-p_i)/p_i]}{p_i^2n^2}\right]. \tag{38}$$

This is only a very rough estimation of an upper limit and surely a better approximation can be found.

5. The generalized Baker's map

The generalized Baker's map was introduced in [7] as an analytically treatable but nontrivial sample for a multifractal. It is defined by

$$x_{n+1} = \begin{cases} \lambda_a x_n & y_n < a \\ \frac{1}{2} + \lambda_b x_n & y_n > a \end{cases}$$

$$y_{n+1} = \begin{cases} (1/a) y_n & y_n < a \\ (1/(1-a))(y_n - a) & y_n > a \end{cases}$$

$$x_n \in [0, 1] \qquad y_n \in [0, 1]. \tag{39}$$

In [7] an expression for the $f(\alpha)$-spectrum of its attractor is given by:

$$f(\alpha) = 1 + \frac{(1-\kappa)\log(1-\kappa) + \kappa \log \kappa}{(1-\kappa)\log \lambda_a + \kappa \log \lambda_b}$$

$$\kappa = \frac{\log a - (\alpha - 1)\log \lambda_a}{(\alpha - 1)\log(\lambda_b/\lambda_a) + \log(a/b)}$$

$$b = 1 - a.$$

To find explicit values for the dimensions we choose arbitrarily

$$\lambda_a = \lambda_b = \tfrac{1}{5} \qquad a = \tfrac{2}{5} \qquad b = \tfrac{3}{5}$$

so that

$$f(\alpha) = \frac{(1-\kappa)\log(1-\kappa) + \kappa \log \kappa}{\log \frac{2}{3}}$$

$$\kappa = \frac{\log \kappa - \log(1-\kappa)}{\log \frac{1}{5}}.$$

By a Legendre transformation

$$(1-q) D_q = q\alpha_q - f(\alpha_q) \tag{40}$$

with

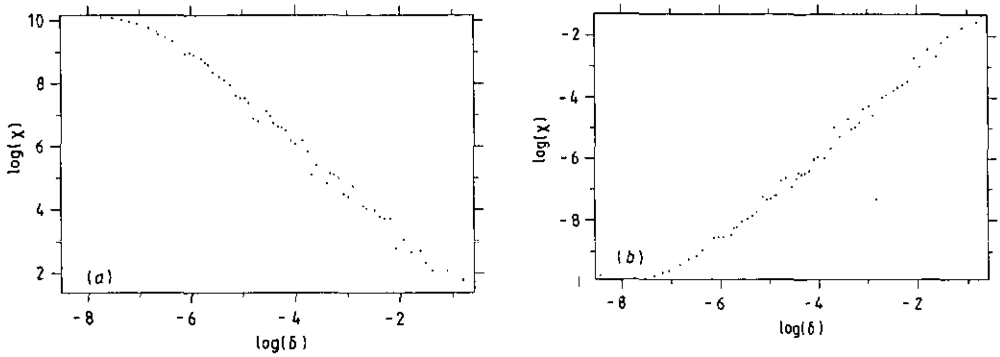
$$q = \frac{\partial f(\alpha)}{\partial \alpha} \tag{41}$$

$$\alpha_q = \alpha \big|_{\partial f(\alpha)/\partial \alpha = q} \tag{42}$$

we get the Rényi dimensions. Using the specified special parameters we can determine from (40)-(42) the dimensions

$$D_{-1} = 1.4434 \qquad D_0 = 1.4307 \qquad D_2 = 1.4063.$$

To compare experimental results to these values we applied the box procedure to the attractor of the generalized Baker's map. Figures 1(a, b, c) show typical plots one gets by this algorithm. It is clearly seen that the data scatter more or less around a trend. It is a known characteristic of this procedure that for small δ the points are closer at the trend than for larger δ. The error bars mark the measured points $\pm\Delta\chi^q$ where the standard deviation is obtained using (26). This equation applied to the usual box algorithm obviously underestimates the existing errors. Indeed it describes a variation



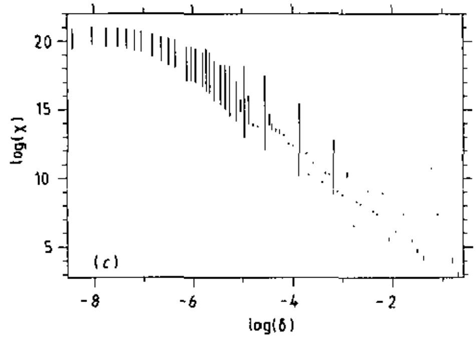


Figure 1. Analysis of the generalized Baker's map by the ordinary box algorithm with $q=0$ (a), $q=2$ (b) and $q=-1$ (c). The very small error bars in plots (a) and (b) mark the measured points plus or minus the calculated standard deviation.

of the time series but in the used algorithm there is another degree of freedom: the translation of the used lattice which causes a variation of the p_i. In table 1 both variations (due to finite length of the used-time series and to the translation of the lattice used) are summarized at different box sizes. The calculated variance is similar to that which we get by varying the time series but it is significantly different from that caused by the translation of the lattice.

The variability of χ^q due to the lattice translation is a consequence of the renunciation of the optimization in (3) and (4) which cannot be realized in numerical evaluations from time series. But it is possible to reduce the variability by optimizing the box algorithm. A procedure that is capable of reducing the fluctuations especially for large δ will be described in the next section.

6. The modified algorithm

In the partition function formalism, covers are regarded that consist of sets with aribtrary small diameters l_i. There is no sense in doing so if we model the attractor by a finite number of points. This, in both the ordinary and the modified box algorithm (OBS, MBA), is handled by regarding covers consisting of sets of equal diameters $l_i \equiv l (\equiv \delta)$. In the OBA those are square boxes out of a lattice and no optimization such as in (3) and (4) is performed. In order to improve the estimation of the χ^q, the MBA starts with a larger family (which we call the prime cover) which contains all circles centred in a point out of the time series with diameter δ. For relevant δ these sets are

Table 1. Here at some values of δ the calculated standard deviation is compared with that from the variation of the time series and that from random translation of the lattice. δ denotes the box size, $\Delta\chi_t^q$ the standard deviation of χ^q from random translation of the box lattice, $\Delta\chi_{tc}^q$ the mean of the calculated standard deviation at random translation, $\Delta\chi_s^q$ the standard deviation from variation of the time series and $\Delta\chi_{sc}^q$ the mean of calculated standard deviations at variation of the time series. Note that at $q=-1$ the calculated standard deviations are just upper limits.

δ	$\Delta\chi_t^{-1}$	$\Delta\chi_{tc}^{-1}$	$\Delta\chi_s^{-1}$	$\Delta\chi_{sc}^{-1}$
1/3	20.35	7.194	0.3740	4.900
1/4	239.9	24.55	0.3609	16.74
1/6	308.9	32.75	1.975	26.10
1/8	1583	91.70	7.123	70.28
1/12	1112	82.04	6.813	69.54
δ	$\Delta\chi_t^{0}$	$\Delta\chi_{tc}^{0}$	$\Delta\chi_s^{0}$	$\Delta\chi_{sc}^{0}$
1/3	0.2421	0.0000	0.0000	0.0000
1/4	0.4227	0.0000	0.0000	0.0000
1/6	0.4285	0.0000	0.0000	0.0000
1/8	0.9682	0.0000	0.0000	0.0000
1/12	0.6960	0.0000	0.0000	0.0000
δ	$\Delta\chi_t^{2}\times 10^4$	$\Delta\chi_{tc}^{2}\times 10^4$	$\Delta\chi_s^{2}\times 10^4$	$\Delta\chi_{sc}^{2}\times 10^4$
1/3	129.6	14.31	11.68	141.0
1/4	25.73	5.277	2.954	5.266
1/6	17.97	5.390	3.612	5.602
1/8	1.834	1.435	0.469	1.445
1/12	2.616	1.400	0.640	1.430

clearly not disjoint and we have to build a subfamily of the prime cover which still is a cover of the attractor and we have to make the sets in it disjoint by subtracting each intersection of several used circles from all but one of them. Simultaneously we want to perform an optimization of the moments according to (3) and (4). For $q>0$ the χ^q are dominated by the fullest boxes (or even the fullest box) and in order to perform the optimization of the χ^q we first want the fullest of the used boxes to be as full as possible, then the second fullest to be as full as possible and so on. For $q<0$ the χ^q are dominated by the emptiest box(es) and to perform the minimization of χ^q we want the empty boxes to be as full as possible.

Table 2. The Rényi dimensions measured using the modified box algorithm together with the estimated errors using (14), (45) show for $q=0, 2$ a good agreement with the calculated values. For $q=-1$ the situation is more complicated.

q	D^q, calculated from (40)-(42)	D^q, measured by the modified box algorithm
0	1.4307	1.434 ± 0.009
2	1.4063	1.413 ± 0.008
−1	1.4434	1.391 ± 0.009

To construct the desired cover we start by choosing the fullest circle out of the prime cover, denote the number of points in this circle by k_1^+ and, because the sets of the desired cover must be disjoint, we delete all points out of the chosen circle from the other sets. Now we choose the fullest of the remaining sets (regarding only the remaining points, not the deleted ones), denote the number of remaining points in it by k_2^+ and delete all points in it from the remaining circles. We continue the same way until all remaining circles are empty. This way we get a cover C^+ of the time series consisting of disjoint sets. For $q \geqslant 0$ we evaluate χ^q as

$$\chi^q = \sum_i \left(\frac{k_i^+}{n}\right)^q = n^{-q} \sum_i (k_i^+)^q. \tag{43}$$

To evaluate χ^q for $q<0$ we regard circles out of C^+ but refill them with all points they contained in the prime cover. We choose the emptiest of those circles, denote the number of points in it by k_1^- and delete all points in it from the other circles. We continue this until no circle out of C^+ remains or all remaining circles are empty. The χ^q with $q<0$ are evaluated as

$$\chi^q = n^{-q} \sum_i (k_i^-)^q. \tag{44}$$

Figure 2 shows typical plots using this modified count algorithm. Here the same time series of 30 000 points was analysed at about the same values of δ as in figure 1. The error bars in the plots in figure 2 mark the measured points plus minus the calculated standard deviation and it can be easily seen that there is a region of an approximately linear behaviour with an overlaying smooth oscillation which is caused by the fact that the self-similarity of the multifractal depends on discrete increasing

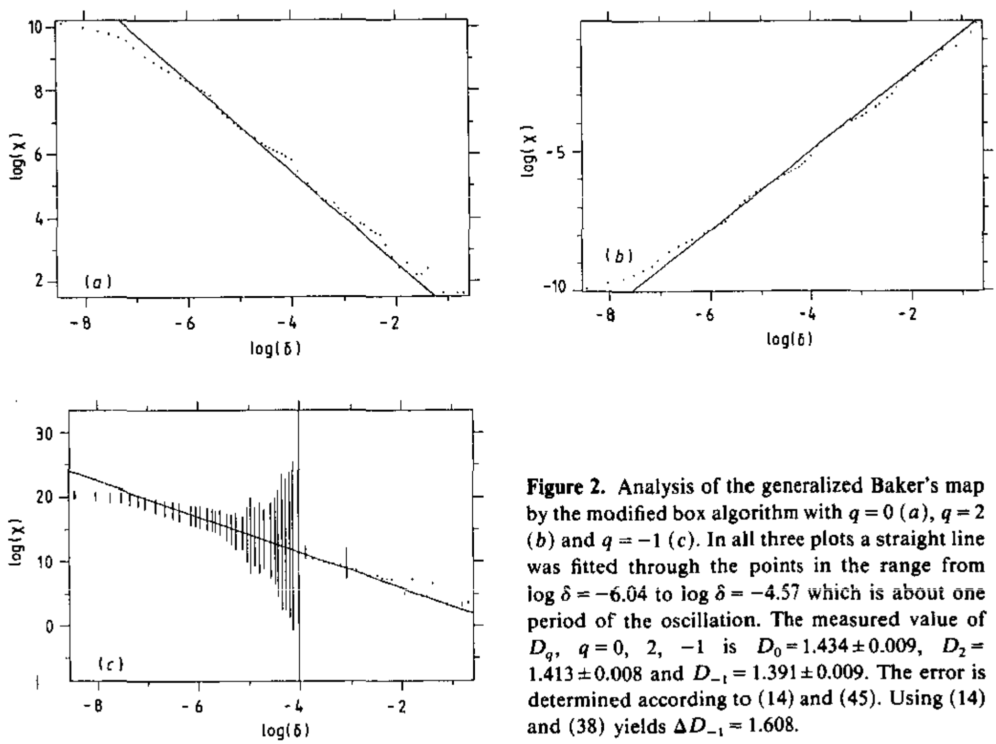


Figure 2. Analysis of the generalized Baker's map by the modified box algorithm with $q=0$ (a), $q=2$ (b) and $q=-1$ (c). In all three plots a straight line was fitted through the points in the range from $\log\delta=-6.04$ to $\log\delta=-4.57$ which is about one period of the oscillation. The measured value of D_q, $q=0$, 2, -1 is $D_0=1.434\pm0.009$, $D_2=1.413\pm0.008$ and $D_{-1}=1.391\pm0.009$. The error is determined according to (14) and (45). Using (14) and (38) yields $\Delta D_{-1}=1.608$.

factors x^r, $r \in \mathbb{N}$. The logarithm of x is the wavelength of the oscillation. Figure 1(a) shows a plot for $q = -1$ and here (26) seems to yield too great error bars at some points. In the power law regime the uncertainty of the points now obviously is very small. Nevertheless the oscillation causes an imprecision in the evaluation of the Rényi dimension and the existing errors would be underestimated if we used the calculated standard deviations by (14) setting $\Delta y_i = \Delta\chi^q(\delta_i)/\chi^q(\delta_i)$. We get a more realistic error if we assume the $\Delta y(\log \delta_i)$ to be equal and estimate them as

$$\Delta y = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (y_i - f(\log \delta_i))^2} \tag{45}$$

with $f(\log \delta)$ as the fitted line. To avoid additional errors the fitted region length must be an integer multiple of the oscillation period. Our fits in figure 2 use about one period of oscillation and for $q = 0$ and $q = 2$ the errors are well estimated while for $q = -1$ the error we get by (14) using (45) is too small (factor >5); on the other hand the error by (14) using (38) is too great (factor ≈ 80). The whole region of approximate power law scaling is only a little longer than used for our fits and shifting the fit range more to the left or more to the right rapidly decreases the measured values of D_q.

7. Discussion

In sections 3 and 4 formulas are derived which describe the statistical error of χ^q due to the finite number of points in a time series. This error is expressed in terms of the p_i which we only know from the analysis whose error we want to describe. Therefore we can only make use of (26) if there are many filled boxes and if each of them contains many points. This is satisfied in a certain range which fortunately is about the same where the power law behaviour is fulfilled. Here the error bars due to (26) can be a good estimation to fix the borders and check the quality of the linear region in the $\log \delta$-$\log \chi^q$ plot. They also allow us to identify runaway points which sometimes occur in the modified algorithm. Then the Rényi dimensions can be measured with a precision that is for positive q well estimated by (14) using (45).

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